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1980 J. Phys. A: Math. Gen. 13 3347

(http://iopscience.iop.org/0305-4470/13/11/008)

J. Phys. A: Math. Gen. **13** (1980) 3347–3355. Printed in Great Britain

k-particle Green function diagrams and the restriction of irreducible representations of a group to its subgroups

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Received 10 January 1980, in final form 30 April 1980

Abstract. Generalisations of Wigner's formulae relating to the counting problem when one considers the restriction of the irreducible representations of a group to its subgroups are obtained in the context of the symmetric group. The usefulness of these in the enumeration problem of many-body diagrams is described.

1. Introduction

The concept of a subclass of a given group was first introduced by Wigner (1971) to study the problem of the restriction of an irreduclble representation of a group to a 'subgroup'. Recently Rosensteel *et al* (1975), Ihrig *et al* (1976) and Wise and Trainor (1978) have classified the k-particle Green functions of many-body perturbation theory with the help of an equivalence relation defined over the elements of the symmetric group s_{2n+k}. This equivalence relation can be identified with the subclass equivalence relation of Wigner in the case of single-particle Green functions and with the double-class equivalence relation in the case of k-particle Green functions, $k \geqslant 1$ (Hasselbarth *et al* 1976). Rosensteel *et al* solved the enumeration problem of the k-particle many-body diagrams using essentially the combinatorial properties of the symmetric group. In this paper we intend to show that one can use the results on group representation obtained by Wigner (1971) to solve the same enumeration problem. For this purpose we first extend Wigner's result using double classes which are a generalisation of subclasses. Finally, starting from the formula for the number of subclasses as given by Wigner, we derive the formula of Rosensteel *et al* (1975) dealing with the enumeration of k-particle many-body diagrams.

The plan of the paper is as follows. In § 2 we generalise the result II of Wigner (1971). In the same section we introduce the many-body enumeration problem and write the formula of Rosensteel *et al* (1975) in representation theoretic language. Using this we derive the combinatorial formula of Rosensteel *et al* (1975). In § 3 we generalise Wigner's formula III.

2. Green diagrams and S_{2n+k}

In order to understand the choice of certain equivalence classes mentioned below we describe briefly the formulation of k-particle Green diagrams by Wise and Trainor (1978). According to them, every nth-order diagram in many-body perturbation

0305-4470/80/113347+09\$01.50

theory can be described by an ordered triple (V, p, τ), etc, where V represents the set of $2n+k$ vertices, $V=\{1, 2, \ldots, 2n, 2n+1, \ldots, 2n+k\}$, $1, 2, \ldots, 2n$ are the internal vertices of the diagram and $2n+1, \ldots, 2n+k$ refer to the external vertices. Further, $p \in s_{2n+k}$ is any permutation of these vertices $(p\colon V \to V)$ and τ is a product of n disjoint transpositions $(\tau\colon V \to V)$ which interchanges distinct pairs of internal vertices but leaves external vertices unaltered.

Given an ordered triple (V, p, τ) one can associate with it a unique nth-order k-particle Green diagram as follows. If p maps an internal vertex i into an internal vertex $j=p(i)$ then a directed line is drawn from i to j. If p also maps the internal vertex l into the external vertex $2n+\mu$, $\mu \in \{1, 2, \ldots, k\}$, and external vertex $2n+\nu$ with $\nu \in \{1, 2, \ldots, k\}$ into the internal vertex h, then external Green lines are drawn from l and inwards to h respectively. Finally, if p maps one external vertex into another external vertex, a free directed Green line is drawn between them. The transpositions composing τ are represented by interaction lines joining the n pairs of internal vertices. Thus the equivalence classes of permutations $p \in s_{2n+k}$ can be linked to the equivalence classes of the corresponding diagrams themselves.

2.1. *Restriction of irreducible representations of a group to its subgroup—a generalisation*

In order to deal with the irreducible representations of s_{2n+k} we adopt the notation of Wigner (1971). Thus, we shall denote the elements of the full group s_{2n+k} by capital letters $P, Q, R, S, \ldots$ and those of the subgroups by the Greek letters $\sigma, \rho, \ldots$. The irreducible representations of the full group s_{2n+k} will be denoted by D with suitable indices and those of the subgroup by d. The indices $J, K, \ldots$ will serve to label the different irreducible representations of the full group while $j, k, \ldots$ will refer to those of the subgroup. Finally, the symbol (J, j) denotes the number of times the irreducible representation D^J of the full group contains the irreducible representation d^j of the subgroup, if the former is restricted to the subgroup. Throughout this paper we are interested in the subgroup $c(\tau_0) \wedge s_k^{\text{ext}}$ of s_{2n+k} (Wise and Trainor 1978) where $c(\tau_0)$ is the centraliser of the element $\tau_0 \in s_{2n+k}$. τ_0 is given by

$$\tau_0 = (12)(34)\ldots(2n-1\,2n)(2n+1)\ldots(2n+k) \in s_{2n+k}.$$

The symbols contained in the two-cycles label the internal vertices of the Green functions and those contained in the one-cycles stand for the external particles. s_k^{ext} is the symmetric group of degree k on the k external symbols. The wedge $\wedge$ stands for the semidirect product (Altman 1977).

Consider the direct product group $s_{2n+k} \times s_{2n+k}\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}$ is a subgroup of it. The elements of $c(\tau_0) \wedge s_k^{\text{ext}}$ are of the form $\sigma\rho$ where $\sigma \in c(\tau_0)$ and $\rho \in s_k^{\text{ext}}$. Therefore the elements of $\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}$ are of the form $(\sigma\rho, \sigma')$. We consider a special subgroup of this group whose elements are of the form $(\sigma\rho, \sigma)$. We denote this subgroup by $\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}\mathrm{d}$. We say that two elements $P, Q \in s_{2n+k}$ belong to the same double class of s_{2n+k} with respect to the subgroup $\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}\mathrm{d}$ if and only if

$$\sigma\rho P \sigma^{-1} = Q \qquad \text{for some } \sigma\rho \in c(\tau_0) \wedge s_k^{\text{ext}} \text{ and } \sigma \in c(\tau_0). \tag{1}$$

Seen in this way, it is clear that the k-particle Green function equivalence classes of Wise and Trainor (1978) are the double classes defined above.

In the following we will derive a formula for the number of double classes defined above. To this end we proceed as follows. The elements of the regular representation

D of the full group s_{2n+k} are given by

$$D(P)_{Q,R} = \delta_{Q,PR} \qquad P, Q, R \in s_{2n+k} \tag{2}$$

where $\delta_{A,B} = 0$ unless the elements A and B are identical.

The rows and columns of the regular representation matrices of s_{2n+k} are labelled by the group elements themselves.

Considering the regular representation defined above, Wigner (1971) has shown that the elements of the matrices A which satisfy the condition

$$D(P)A = AD(P) \qquad \forall P \in s_{2n+k} \tag{3}$$

are given by

$$A_{Q,R} = a_{Q^{-1}R} \tag{4}$$

and, similarly, the elements of the matrices B which obey the condition

$$AB = BA \qquad \forall A \tag{5}$$

are given by

$$B_{Q,R} = b_{QR^{-1}}. \tag{6}$$

We consider now the sets of matrices which intertwine the restriction of the regular representation of s_{2n+k} *to* $c(\tau_0) \wedge s_k^{\text{ext}}$ *and* $c(\tau_0)$. The set of matrices $\{D(\sigma\rho)|\sigma\rho \in c(\tau_0) \wedge s_k^{\text{ext}}\}$ belongs to the regular representation of s_{2n+k}. It forms a reducible representation of $c(\tau_0) \wedge s_k^{\text{ext}}$. Similarly, the subset $\{D(\sigma)|\sigma \in c(\tau_0)\}$ forms a reducible representation of $c(\tau_0)$. This last representation is a homomorphism from $c(\tau_0) \wedge s_k^{\text{ext}}$ to $c(\tau_0)$. We denote the subset of matrices B which intertwine the reducible representations $\{D(\sigma\rho)|\sigma\rho \in c(\tau_0) \wedge s_k^{\text{ext}}\}$ and $\{D(\sigma)|\sigma \in c(\tau_0)\}$ with each other by the letter C. Therefore the matrices C satisfy the condition

$$D(\sigma\rho)C = CD(\sigma) \qquad \forall \sigma \in c(\tau_0), \rho \in s_k^{\text{ext}}. \tag{7}$$

We wish to know the number of linearly-independent matrices C which satisfy the above condition. For this purpose we write the matrices $D(\sigma\rho)$ and $D(\sigma)$ in their fully reduced form in terms of the irreducible constituents of the subgroup $c(\tau_0) \wedge s_k^{\text{ext}}$ and make use of Schur's lemma to conclude that

$$cd^j(\sigma\rho)c^{-1} = d^j(\sigma). \tag{8}$$

This implies that

$$d^j(\sigma\rho) = d^j(\sigma) \qquad \forall \sigma \in c(\tau_0), \rho \in s_k^{\text{ext}}. \tag{9}$$

Therefore, following Wigner (1971), we obtain for the dimension d_C of the set of matrices C

$$d_C = \sum_{J,j} (J, j)^2, \tag{10}$$

where, as in the case of subclasses, J runs through all the irreducible representations of s_{2n+k}, each taken only once, and j runs through all the representations d^j for which

$$d^j(\sigma\rho) = d^j(\sigma) \qquad \forall \sigma \in c(\tau_0), \rho \in s_k^{\text{ext}}.$$

We indicate this restriction in j by putting an asterisk on the summation sign:

$$d_C = \sum_{J,j}^{*} (J, j)^2. \tag{11}$$

We now connect the above formula to D_c, the number of double classes of s_{2n+k} with respect to $\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}$d, in the following manner. Consider the equation

$$D(\sigma\rho)C = CD(\sigma) \qquad \forall\sigma \in c(\tau_0),\, \rho \in s_k^{\text{ext}} \tag{12}$$

where $D(\sigma\rho)$ and $D(\sigma)$ have either one or zero as their elements. From the above equation we get

$$(D(\sigma\rho)C)_{Q,S} = (CD(\sigma))_{Q,S} \tag{13}$$

or

$$\sum_R D_{Q,R}(\sigma\rho)C_{R,S} = \sum_R C_{Q,R}D_{R,S}(\sigma) \tag{14}$$

that is,

$$\sum_R \delta_{Q,R}(\sigma\rho)C_{R,S} = \sum_R C_{Q,R}\delta_{R,S}(\sigma). \tag{15}$$

Remembering that the C matrices are a subset of the B matrices, we replace $C_{R,S}$ and $C_{Q,R}$ by $C_{RS^{-1}}$ and $C_{QR^{-1}}$ respectively. Thus

$$\sum_R \delta_{Q,R}(\sigma\rho)C_{RS^{-1}} = \sum_R C_{QR^{-1}}\delta_{R,S}(\sigma) \qquad \forall\sigma \in c(\tau_0),\, \rho \in s_k^{\text{ext}}. \tag{16}$$

Thus the condition which the elements of the matrices C have to satisfy is

$$C_{(\sigma\rho)^{-1}QS^{-1}} = C_{QS^{-1}\sigma^{-1}}. \tag{17}$$

This is possible if and only if the elements of C labelled by elements belonging to the same double class of s_{2n+k} with respect to $\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}$d are equal to each other. From the above it is clear that

$$d_C = D_c. \tag{18}$$

Formula (18) corresponds to Wigner's formula II.

If in the above $s_k^{\text{ext}} = e$, the identity element, then the double classes will become subclasses and in the formula for the number of subclasses the symbol j runs through all the irreducible representations of $c(\tau_0)$. When we specialise in this way all the results of Wigner (1971) for the subclasses are applicable here also.

In the following we derive the formula of Rosensteel *et al* (1975) for the number of subclasses from our formula for the same. We have just seen that the number of subclasses of s_{2n+1} with respect to $c(\tau_0)$, N_s say, is given by

$$N_s = \sum_{J,j} (J, j)^2, \tag{19}$$

where we emphasise that the symbol j runs through all the irreducible constituents of $c(\tau_0)$.

To proceed further we write the intertwining numbers (J, j) in terms of the components of the character vectors of the irreducible representations of s_{2n+1} and of $c(\tau_0)$ in the following well known form (Altman 1977):

$$(J, j) = \frac{1}{{}^{0}c(\tau_0)} \sum_l \sum_{l'(l)} c_{l'(l)}\phi_{l'(l)}^{(j)}\chi_l^{(J)*} \tag{20}$$

where l is the label of the classes of s_{2n+1} and the summation extends over all classes of s_{2n+1}. $l'(l)$ is the label of a subclass of s_{2n+1} obtained from the class l of s_{2n+1} and contained as a class in $c(\tau_0)$. The summation over $l'(l)$ extends over all such subclasses.

$c_{l'(l)}$ is the size of the subclass $l'(l)$ of s_{2n+1} contained in $c(\tau_0)$. $\phi^{(j)}_{l'(l)}$ is the $l'(l)$th component of the character vector $\phi^{(j)}$ of the jth irreducible representation of $c(\tau_0)$. $\chi^{(J)}_l$ is the lth component of the character vector $\chi^{(J)}$ of the irreducible representation J of s_{2n+1}. The symbol $*$ denotes complex conjugation. Finally, ${}^0c(\tau_0)$ is the order of $c(\tau_0)$. Squaring the expression for the intertwining number (J, j), we obtain

$$(J, j)^2 = \frac{1}{({}^0c(\tau_0))^2} \sum_{l_1} \sum_{l_2} \sum_{l'(l_1)} \sum_{l'(l_2)} c_{l'(l_1)} c_{l'(l_2)} \phi^{*(j)}_{l'(l_1)} \phi^{(j)}_{l'(l_2)} \chi^{*(J)}_{l_1} \chi^{(J)}_{l_2} \tag{21}$$

where the symbols l_1 and l_2 representing two different classes of s_{2n+1} take care of the cross terms. We now sum both sides of (21) over all J of s_{2n+1} and over all j of $c(\tau_0)$. On the right-hand side we perform this summation before the summation over l_1, l_2, $l'(l_1)$ and $l'(l_2)$. Therefore

$$N_s = \frac{1}{({}^0c(\tau_0))^2} \sum_{l_1} \sum_{l_2} \sum_{l'(l_1)} \sum_{l'(l_2)} \sum_{J} \sum_{j} c_{l'(l_1)} c_{l'(l_2)} \phi^{*(j)}_{l'(l_1)} \phi^{(j)}_{l'(l_2)} \chi^{*(J)}_{l_1} \chi^{(J)}_{l_2}. \tag{22}$$

In evaluating the summations over J and j we make use of the well known orthogonality relations for irreducible characters (Altman 1977):

$$\sum_{J=1}^{p} \chi^{*(J)}_{l_1} \chi^{(J)}_{l_2} = \frac{{}^0s_{2n+1}}{(c_{l_1 l_2})^{1/2}} \delta_{l_1 l_2} \tag{23}$$

where

$$\delta_{l_1 l_2} = 1 \qquad \text{if } l_1 = l_2$$
$$= 0 \qquad \text{otherwise}$$

and p indicates the number of irreducible representations of s_{2n+1}. ${}^0s_{2n+1}$ is the order of s_{2n+1}.

$$\sum_{j=1}^{s} \phi^{*(j)}_{l'(l_1)} \phi^{(j)}_{l'(l_2)} = \frac{{}^0c(\tau_0)}{(c_{l'(l_1)} c_{l'(l_2)})^{1/2}} \delta_{l_1 l_2}. \tag{24}$$

Therefore

$$N_s = \frac{1}{({}^0c(\tau_0))^2} \sum_{l} \sum_{l'(l)} c_{l'(l)} {}^0c(\tau_0) {}^0s_{2n+1}/c_l. \tag{25}$$

In obtaining the above expression from its predecessor we have cancelled ${}^0c(\tau_0)$ from the numerator and the denominator and have dropped the suffix 1 of the symbol l_1 as it is not required now.

But from a well known counting principle in group theory (Herstein 1964) we know that

$${}^0s_{2n+1}/{}^0c_l = {}^0c(\tau_l) \tag{26}$$

where ${}^0c(\tau_l)$ represents the order of the centraliser of the representative element τ_l of the equivalence class l of s_{2n+1}. Substituting this in the last expression for $\Sigma_{J,j}\ (J, j)^2$ we obtain

$$\sum_{J,j} (J, j)^2 = \frac{1}{{}^0c(\tau_0)} \sum_{l} \sum_{l'(l)} c_{l'(l)} {}^0c(\tau_l). \tag{27}$$

Finally, remembering the fact that $c_{l'(l)} = 0$ unless $c_l \cap c(\tau_0) \neq \varnothing$ the null set (c_l is here the lth class of s_{2n+1} and not its size), we convert the summation over l and $l'(l)$ into a summation over the elements of $c(\tau_0)$ and obtain

$$\sum_{J,j}(J,j)^2=\frac{1}{{}^0c(\tau_0)}\sum_{\tau\in c(\tau_0)}{}^0c(\tau). \tag{28}$$

We immediately identify the right-hand side of the above equation as the formula for the number of subclasses of s_{2n+1} with respect to $c(\tau_0)$ as derived by Rosensteel *et al* (1975).

2.2. The algebra of the subclasses

We now turn our attention to the algebra of the subclasses of s_{2n+1} with respect to the subgroup $c(\tau_0)$ and examine the condition that Wigner obtained for (J, j) to be either 1 or 0 for any pair of symbols J and j. It turns out that for the group–subgroup pair s_{2n+1} and $c(\tau_0)$ there will be at least one pair of symbols J, j such that the corresponding intertwining number (J, j) is greater than 1. The proof is as follows.

Proof. We prove the above statement by showing that there exist subclasses which do not commute with each other. This automatically implies that $(J, j) > 1$ for at least one pair J, j.

Consider the algebra of subclasses of s_{2n+1} with respect to $c(\tau_0)$. According to the formulation of Rosensteel *et al* (1975) each subclass of s_{2n+1} represents a set of topologically equivalent many-body diagrams. Diagrams belonging to different subclasses are topologically inequivalent to each other. Therefore, in diagrammatic language, each subclass is either connected or disconnected. This means that diagrams occurring in a connected subclass are all connected and those occurring in a disconnected subclass are all disconnected. Consider the set of elements

$$s_1s_2s_1^{-1}. \tag{29}$$

Take, for s_1, the subclass in which the element $(0, 1, 2, \ldots, 2n)$ occurs. s_1^{-1} stands for the subclass formed by taking the inverses of the elements of s_1. For s_2 take that subclass in which the element $(0)\ (123\ldots 2n)$ occurs. The set $s_1s_2s_1^{-1}$ involves, therefore, products of the form

$$(0123\ldots 2n)(0)(12\ldots 2n)(02n\ldots 1)=(012\ldots 2n-1)(2n).$$

The resulting element has the same cyclic structure as that of $(0)\ (12,\ldots 2n)$ but in it the one-cycle contains the symbol $2n$ instead of '0'.

From the above two things are clear:

(1) s_1 and s_2 do not commute;

(2) if a disconnected subclass is conjugated with a connected one the result will also involve connected subclasses.

Now consider the case when s_2 is the subclass consisting of the element $(0)\ (12\ldots 2n)$ and s_1 is the one containing $(0)\ (12)\ldots(2n-1\ 2n)$ only. We see that $s_1s_2s_1^{-1}$ has elements of the form

$$\begin{aligned}&(0)(12)(34)\ldots(2n-1\ 2n)(0)(12\ldots 2n)(0)(12)\ldots(2n-1\ 2n)\\ &\qquad=(0)(1436\ldots 2n-1\ 2).\end{aligned} \tag{30}$$

The following two points emerge from (30):

(3) s_1 and s_2 do not commute with each other;

(4) the commutator of s_1 and s_2 consists of only disconnected subclasses.

From points (1), (2), (3) and (4) above we see that there exist at least two pairs of subclasses whose commutators are not identical. Therefore the number of linearly independent commutators of the algebra of subclasses of s_{2n+1} with respect to $c(\tau_0)$ is never zero. Hence there exists at least one (J, j) which is different from both zero and one.

The above discussion demonstrates the power of Wigner's method in deciding whether $(J, j) = 0$ or 1 without actually calculating them through a knowledge of the character tables of s_{2n+1} and of $c(\tau_0)$, which may be quite tedious.

Finally, we observe that one can also prove by extending the previous arguments to s_{2n+k} and the corresponding subgroup $c(\tau_0) \wedge s_k^{\text{ext}}$ that the number of linearly independent commutators of the algebra of double classes of s_{2n+k} with respect to $\{(c(\tau_0) \wedge s_k^{\text{ext}}) \times c(\tau_0)\}$ is not zero. But this fact does not seem to have any bearing on the value of the intertwining number (J, j). The reason for this is the fact that one cannot formulate Wigner's condition in the context of double classes solely in terms of the linearly independent commutators of s_{2n+k} with respect to $c(\tau_0) \wedge s_k^{\text{ext}}$. This fact will be discussed below.

2.3. *Example*

As an example of the preceding discussion, we consider the group $s_5 = s_{2\times2+1}$ and its subgroup $c(\tau_0)$, $\tau_0 = (0)(12)(34) \in s_5$. This means that we are concerned with single-particle Green functions and their equivalence classes under the action of $c(\tau_0)$. Under the action of $c(\tau_0)$, s_5 splits into 22 subclasses. This number is calculated using the formulae of both Wigner and Rosensteel *et al* separately. All four irreducible representations of $c(\tau_0)$ occur in the seven irreducible representations of s_5 each once or zero times only, except for one which occurs twice. Hence the subclasses are not commutative. There are altogether three linearly independent commutators of the algebra of subclasses. This number is calculated from Wigner's formula. These are, for example, the subclasses containing the elements (01234) and (0123)(4), those containing (01234) and (0)(1234), and the ones containing (0134)(2) and (0)(1234).

In the following we try to obtain, for double classes, a condition similar to the one obtained by Wigner (1971) for the commutativity of the subclasses in terms of the values of the intertwining numbers (J, j). For this purpose we consider the following.

3. The centre of the set of C

The centre of the set of matrices C, by definition, consists of those matrices Z which commute with all matrices A, with all the matrices C and intertwines $D(\sigma\rho)$ with $D(\sigma)$ for all $\sigma \in c(\tau_0)$ and $\rho \in s_k^{\text{ext}}$.

Adopting the same procedure as that of Wigner (1971), one can show that

$$\sum_{J,j}{}^{*} (J, j)^{\epsilon} = \text{dimension of } Z = d_z, \tag{31}$$

$(J, j)^{\epsilon}$ being one if D^J contains j, zero otherwise.

We denote the matrices belonging to the centre Z of the set of matrices C by Z. Since the Z are members of the set B, we can write

$$Z_{Q,R} = z_{QR^{-1}} \tag{32}$$

and since they are also members of the set C the elements of the matrices Z which are labelled by the elements of a double class are all equal to each other i.e.

$$Z_P = z_{\sigma\rho P\sigma^{-1}}. \tag{33}$$

The additional condition on the Z is that they commute with any matrix C, i.e.

$$\sum_R c_{QR^{-1}} z_{RS^{-1}} = \sum_R z_{QR^{-1}} c_{RS^{-1}} \tag{34}$$

must be valid for any c_P which is a double-class function. Writing P^{-1} for QR^{-1} on the left side and for RS^{-1} on the right side (this is possible since R is a running index on both sides) this assumes the form

$$\sum_P C_{P^{-1}} Z_{PQS^{-1}} = \sum_P C_{P^{-1}} z_{QS^{-1}P}. \tag{35}$$

The above equations will be satisfied if they are satisfied for all C which assume the value one on double class, zero on all others, and if this is true for all double classes. Moreover, if $U = \sigma\rho V\sigma^{-1}$ then the inverses of U and V are related to each other by the equation $U^{-1} = \sigma V^{-1}\rho^{-1}\sigma^{-1}$, which is an equivalence relation on s_{2n+k} with respect to the subgroup $\{c(\tau_0)\times(c(\tau_0)\wedge s_k^{\text{ext}})\}$d. The equivalence classes are again double classes but now with respect to the subgroup mentioned above. Hence, the P^{-1} form a double class with respect to $\{c(\tau_0)\times(c(\tau_0)\wedge s_k^{\text{ext}})\}$d if the P do with respect to $\{(c(\tau_0)\wedge s_k^{\text{ext}})\times c(\tau_0)\}$d.

From the preceding discussion it follows that

$$\sum_{Pcd^{-1}} z_{PR} = \sum_{Pcd^{-1}} z_{RP}. \tag{36}$$

In the above QS^{-1} has been replaced by R, the summation over P is to be extended over one double class which is made up of the inverses of the elements P^{-1}. The above equation is true no matter over which double class the summation takes place as we know that Z itself is a double-class function.

Because of the above reasoning we can write

$$\sum_\sigma \sum_{Pcd^{-1}} z_{\sigma PR\sigma^{-1}} = \sum_\sigma \sum_{Pcd^{-1}} z_{\sigma RP\sigma^{-1}} \tag{37}$$

where the summation over σ is to be extended over the subgroup $c(\tau_0)$.

The above equation can be written as

$$\sum_\sigma \sum_{Pcd^{-1}} z_{\sigma P\sigma^{-1}\sigma R\sigma^{-1}} = \sum_\sigma \sum_{Pcd^{-1}} z_{\sigma R\sigma^{-1}\sigma P\sigma^{-1}}. \tag{38}$$

Replacing $\sigma P\sigma^{-1}$ by Q and $\sigma R\sigma^{-1}$ by T where $Q\subset d^{-1}$ and $T\subset s$ are subclasses of s_{2n+k} with respect to $c(\tau_0)$ we get

$$\sum_{Q\subset d^{-1}} \sum_{T\subset S} z_{QT} = \sum_{Q\subset d^{-1}} \sum_{T\subset s} z_{TQ}. \tag{39}$$

We now consider the product of a double class d^{-1} and a subclass s, that is the product $d^{-1}s$. This product contains entire subclasses of s_{2n+k} with respect to $c(\tau_0)$. This is because, following Wigner's (1971) argument, an element occurs in $d^{-1}s$ and in $\sigma d^{-1}s\sigma^{-1} = \sigma d^{-1}\sigma^{-1}\sigma s\sigma^{-1} = d^{-1}s$ an equal number of times.

Keeping the above point regarding the product of a double class d^{-1} and a subclass s in mind we now go through the same arguments as those of Wigner (1971) and obtain the following formula for d_z:

$$d_z = \sum_{J,j}^{*} (J, j)^{\epsilon} = D_c - L_{[d^{-1},s]} \tag{40}$$

where $L_{[d^{-1},s]}$ is the number of linearly independent commutators of a double class d^{-1} and a subclass s. Combining this with the previous result for the number of double classes we obtain

$$d_c - D_c = L_{[d^{-1},s]}. \tag{41}$$

This corresponds to Wigner's formula III. All the above results reduce to the results obtained by Wigner for the subclasses if we put $s_k^{\text{ext}} = e$, the identity group.

4. Conclusion

In conclusion, we note that from the above discussion there emerge certain points which may be worthwhile to investigate. They are:

(1) the extension of the above generalisation to include any subgroup $G_1 \times G_2$;

(2) the physical meaning of the commutativity or otherwise of the subclasses representing many-body diagrams.

We also observe that the above analysis can also be applied to solve the enumeration problem of the permutation isomers mentioned by Hasselbath *et al* (1977).

Acknowledgments

I wish to express my gratitude to Professor Alladi Ramakrishnan, Director, Matscience for constant encouragement. I must express my sincere thanks to Professor N R Ranganathan for not only suggesting this problem for investigation but also for many helpful discussions on several aspects of the problem. My thanks are also due to Professor K Srinivasa Rao for lending me the reprints of Wigner's papers which form the basis of this investigation. I also wish to convey my thanks to Dr R Jagannathan and Ms S N Uma for several discussions.

References

Altman S L 1977 *Induced Representations in Crystals and Molecules* (New York: Academic)
Hasselbarth W, Ruch D, Kelin J and Seligman T H 1976 *Proc. 5th Int. Colloq., Université de Montreal*
Herstein I N 1964 *Topics in Algebra* (New York: Blaisdell)
Ihrig E, Rosensteel G, Chow H and Trainor L E H 1976 *Proc. R. Soc.* A **348** 339–57
Rosensteel G, Ihrig E and Trainor L E H 1975 *Proc. R. Soc.* A **344** 387–401
Wigner R P 1971 *Proc. R. Soc.* A **322** 181–9
Wise M B and Trainor L E H 1978 *Proc. R. Soc.* A **259** 11–119